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Classification of 2-arc-transitive dihedrants

Shaofei Du [a,1], Aleksander Malnič [b,2], Dragan Marušič [b,c,2]

[a] *Capital Normal University, Bejing 100037, People's Republic of China*
[b] *University of Ljubljana, IMFM, Jadranska 19, 1000 Ljubljana, Slovenia*
[c] *University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia*

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Abstract

A complete classification of 2-arc-transitive *dihedrants*, that is, Cayley graphs of dihedral groups is given, thus completing the study of these graphs initiated by the third author in [D. Marušič, On 2-arc-transitivity of Cayley graphs, J. Combin. Theory Ser. B 87 (2003) 162–196]. The list consists of the following graphs:

(i) cycles C_{2n}, $n \geqslant 3$;
(ii) complete graphs K_{2n}, $n \geqslant 3$;
(iii) complete bipartite graphs $K_{n,n}$, $n \geqslant 3$;
(iv) complete bipartite graphs minus a matching $K_{n,n} - nK_2$, $n \geqslant 3$;
(v) incidence and nonincidence graphs $B(H_{11})$ and $B'(H_{11})$ of the Hadamard design on 11 points;
(vi) incidence and nonincidence graphs $B(\mathrm{PG}(d,q))$ and $B'(\mathrm{PG}(d,q))$, with $d \geqslant 2$ and q a prime power, of projective spaces;
(vii) and an infinite family of regular $\mathbb{Z}_d$-covers K_{q+1}^{2d} of $K_{q+1,q+1} - (q+1)K_2$, where $q \geqslant 3$ is an odd prime power and d is a divisor of $\frac{q-1}{2}$ and $q-1$, respectively, depending on whether $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$, obtained by identifying the vertex set of the base graph with two copies of the projective line $\mathrm{PG}(1,q)$, where the missing matching consists of all pairs of the form $[i,i']$, $i \in \mathrm{PG}(1,q)$, and the edge $[i,j']$ carries trivial voltage if $i = \infty$ or $j = \infty$, and carries voltage $\bar{h} \in \mathbb{Z}_d$, the residue class of $h \in \mathbb{Z}$, if and only if $i - j = \theta^h$, where θ generates the multiplicative group $\mathbb{F}_q^*$ of the Galois field $\mathbb{F}_q$.

E-mail address: dragan.marusic@upr.si (D. Marušič).

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1. Introduction

Classifying mathematical objects, that is, getting a satisfactory understanding of their structure, can almost as a rule be carried out only for certain reasonably restricted classes of objects. For example, obtaining a classification of all 1-arc-transitive graphs is infeasible because this class is simply too rich. The situation seems to be somewhat more promising with s-arc-transitive graphs where $s \geqslant 2$. The strategy for the structural analysis of these graphs, based on taking normal quotients, was first laid out by Praeger (see [36–38]) and can be summarized as follows. Let X be a graph admitting a 2-arc-transitive action of a subgroup $G \leqslant \mathrm{Aut}\, X$, and let K be a nontrivial normal subgroup of G having at least three orbits. Then the quotient projection $X \to X_K$ by the action of K is a regular covering projection, and the quotient graph X_K admits a 2-arc-transitive action of G/K. Repeating this process one eventually ends up with a 2-arc-transitive graph whose every nontrivial normal subgroup of its automorphism group has at most two orbits, that is, the graph is either quasiprimitive or biquasiprimitive, in short, *basic*. (Recall that a graph is *quasiprimitive* if every nontrivial normal subgroup of its automorphism group is transitive, and is *biquasiprimitive* if it has a nontrivial normal subgroup with two orbits but no such subgroup with more than two orbits.) These observations were made first in [35] in a slightly more general context of locally primitive graphs, that is, vertex-transitive graphs with vertex stabilizers acting primitively on the corresponding neighbors' sets (see [39, Theorem 10.2]). They suggest an obvious (at least in principle) strategy leading to a possible classification of 2-arc-transitive graphs, involving the following two steps. *Step 1* would be concerned with obtaining a description of basic graphs, and *Step 2* would then consist in finding all 2-arc-transitive regular covers of these basic graphs.

It is Step 1 that has received most of the attention thus far. In [21], Ivanov and Praeger completed the classification of quasiprimitive graphs of affine type, and Baddeley gave a detailed description of quasiprimitive graphs of twisted wreath type [2]. A similar description of 2-arc-transitive graphs associated with Suzuki groups and Ree groups was obtained by Fang and Praeger [11,12]. Let us also mention a series of more recent papers by Li [25–27], where, among other, a classification of quasiprimitive 2-arc-transitive graphs of odd order and prime power order has been completed. Finally, most recently, based on this approach, finite vertex-primitive 2-arc-regular graphs have been classified [13] and finite 2-arc-transitive Cayley graphs of abelian groups have been determined [28].

As for Step 2, although graph-covering techniques have been around for quite a while now, see for example [7,16,17], classification problems involving covers have not been a central goal, until quite recently. In [9], regular covers of complete graphs whose group of covering transformations is either cyclic or isomorphic to $\mathbb{Z}_p^2$, p a prime, and whose fibre-preserving subgroup of automorphisms acts 2-arc-transitively, were classified. This result has been extended to the case where the group of covering transformations is isomorphic to $\mathbb{Z}_p^3$, p a prime [8].

The main goal of this article is the classification of 2-arc-transitive *dihedrants*, that is, Cayley graphs of dihedral groups, and thus a completion of a particular line of research initiated in [1, Theorem 1.1], where a complete classification of 2-arc-transitive circulants was given, and followed by [31], where a structural reduction theorem for 2-arc-transitive dihedrants was obtained. (We remark that considerable attention has recently also been given to various other research questions about dihedrants, see [14,22–24,34,42].) The method used in [31] as well as in this article differs somewhat from the Praeger normal subgroup reduction method in that the quotienting is not always done with respect to a normal subgroup. The method relies heavily on the classical results on B-groups due to Schur and Wielandt, which says that cyclic groups of composite order and dihedral groups are all B-groups [43, Theorems 25.3 and 25.6]. (Recall that a group G is a *B-group* if every permutation group containing a regular subgroup isomorphic to G is either imprimitive or doubly transitive.) In particular, the automorphism group A of a 2-arc-transitive Cayley graph $X \ncong K_{2n}$ of a dihedral group D_{2n} is necessarily imprimitive. By letting X be nonbasic and by taking the quotient $X_{\mathcal{B}}$ of X relative to a minimal block system $\mathcal{B}$ of A (not necessarily arising from the orbits of some normal subgroup of A), it can be proved that X is a cover of $X_{\mathcal{B}}$, and that the latter is a 2-arc-transitive Cayley graph of a cyclic or dihedral group, thus laying grounds for a reduction that replaces the Praeger normal subgroup reduction in the general setting.

Translated into the language of this article, the main result of [1] says that, first, the only quasiprimitive and biquasiprimitive 2-arc-transitive circulants of valency greater than 2 are, respectively, complete graphs and complete bipartite graphs, and second, the only nonbasic 2-arc-transitive circulants of valency greater than 2 are complete bipartite graphs minus a matching $K_{m,m} - mK_2$, m odd (that is, the canonical double covers of complete graphs K_m). As for the structural reduction theorem given in [31, Theorem 2.1] and its corrected and slightly more general version given in [32, Theorem 1], it implies that a nonbasic 2-arc-transitive dihedrant is necessarily a cyclic regular cover either of a basic graph or of $K_{q+1,q+1} - (q+1)K_2$, where q is an odd prime power.

More precisely, let $n \geqslant 3$ and let $\mathcal{G}$ denote the class of graphs containing complete bipartite graphs $K_{n,n}$, complete bipartite graphs minus a matching $K_{n,n} - nK_2$, incidence and nonincidence graphs $B(H_{11})$ and $B'(H_{11})$ of the Hadamard design on 11 points, and incidence and nonincidence graphs $B(\mathrm{PG}(d,q))$ and $B'(\mathrm{PG}(d,q))$, with $d \geqslant 2$ and q a prime power, of projective spaces; and we let $\mathcal{H}$ denote the class of graphs containing cycles C_{2n}, complete graphs K_{2n}, and graphs K^4_{q+1} obtained as regular $\mathbb{Z}_2$-covers of $K_{q+1,q+1} - (q+1)K_2$, q an odd prime power, by identifying the vertex set of the base graph with two copies of the projective line $\mathrm{PG}(1,q)$ where the missing matching in $K_{q+1,q+1} - (q+1)K_2$ consists of all pairs $[i,i']$, $i \in \mathrm{PG}(1,q)$, and the edge $[i,j']$ carries voltage 1 if $i-j$ is a nonsquare in $\mathbb{F}_q$, and voltage 0 in all other cases. Then [32, Theorem 1] (a slightly more general version of [31, Theorem 2.1]) says that a 2-arc-transitive dihedrant either belongs to $\mathcal{G} \cup \mathcal{H}$ or is a cyclic regular cover of a graph in $\mathcal{G}$ (see below).

Theorem 1.1. *(See [32, Theorem 1].) Let $n \geqslant 3$, and let X be a connected, 2-arc-transitive Cayley graph of a dihedral group $D = D_{2n} = \langle \rho, \tau \mid \rho^n = \tau^2 = (\rho\tau)^2 = 1\rangle$ of order $2n$. Then one of the following occurs*:

(i) *either $X \in \mathcal{G} \cup \mathcal{H}$; or*
(ii) *X is a regular cyclic cover of a graph in $\mathcal{G}$; more precisely: there exists a proper divisor m of n such that the set $\mathcal{B}$ of orbits of $\langle \rho^m\rangle$ is an imprimitivity block system of* $\operatorname{Aut} X$ *relative to which X is a regular $\mathbb{Z}_{n/m}$-cover of $X_{\mathcal{B}}$, the latter being a graph in $\mathcal{G}$ admitting a regular dihedral group $D/\langle \rho^m\rangle$.*

Of the graphs involved in the statement of Theorem 1.1, the only nonobvious examples of 2-arc-transitive dihedrants are the graphs K^4_{q+1}. These graphs were first constructed in [9], but them being dihedrants was proved in [31]. There is a natural generalization of this construction essential to this article. Let q be an odd prime power, and let d be a divisor of $\frac{q-1}{2}$ when $q \equiv 1 \pmod 4$, and divisor of $q-1$ when $q \equiv 3 \pmod 4$, respectively. Further, let $\mathbb{F}_q^* = \langle \theta \rangle$. Then K_{q+1}^{2d} is obtained as a regular $\mathbb{Z}_d$-cover of $K_{q+1,q+1} - (q+1)K_2$ by identifying the vertex set of the base graph with two copies of the projective line $\mathrm{PG}(1,q)$, where the missing matching in $K_{q+1,q+1} - (q+1)K_2$ consists of all pairs $[i, i']$, $i \in \mathrm{PG}(1,q)$, and the edge $[i, j']$ carries trivial voltage if $i = \infty$ or $j = \infty$, and carries voltage $\bar{h} \in \mathbb{Z}_d$, the residue class of $h \in \mathbb{Z}$, if and only if $i - j = \theta^h$.

We may now state the main classification result of this article.

Theorem 1.2. *Let $n \geqslant 3$ and let X be a connected 2-arc-transitive Cayley graph of a dihedral group of order $2n$. Then one of the following occurs*:

(i) *X is a basic graph and is isomorphic to one of the following graphs*: *C_{2n}, n a prime*; *K_{2n}*; *$K_{n,n}$*; *$B(H_{11})$ or $B'(H_{11})$*; *$B(\mathrm{PG}(d,q))$ or $B'(\mathrm{PG}(d,q))$, where $n = (q^d - 1)/(q-1)$, $d \geqslant 2$, and q is a prime power*; *or*

(ii) *X is not a basic graph and either X is isomorphic to $K_{n,n} - nK_2$, or there exists an odd prime power q such that $n = q+1$ and X is isomorphic to K_{q+1}^{2d}, where d is a divisor of $\frac{q-1}{2}$ if $q \equiv 1 \pmod 4$, and a divisor of $q-1$ if $q \equiv 3 \pmod 4$, respectively.*

Let us remark that, translated into group-theoretic language, Theorem 1.2 essentially describes (subject to computing the automorphism groups of graphs K_{q+1}^{2d}) all transitive permutation groups containing a regular dihedral subgroup having point stabilizers acting doubly transitively on some nontrivial connected suborbit, that is, on some suborbit which gives rise to a connected coset graph (sometimes also called orbital graph in the literature).

This article is organized as follows. In Section 2, we introduce special terminology and notation, coset and bi-coset graphs, coverings and lifting automorphisms and some related group-theoretic results. In the next three sections, nonexistence of 2-arc-transitive dihedrants among regular cyclic covers of basic graphs is shown: in Section 3 for the incidence and nonincidence graphs $B(H_{11})$ and $B'(H_{11})$, respectively, of the unique Hadamard design H_{11} on 11 points; in Section 4 for the complete bipartite graphs; and in Section 5 for the incidence and nonincidence graphs $B(\mathrm{PG}(d,q))$ $B'(\mathrm{PG}(d,q))$, respectively, associated with projective spaces $\mathrm{PG}(d,q)$. Since in the reduction, the blocks of the automorphism group of a 2-arc-transitive dihedrant can always be assumed minimal, only nonexistence of 2-arc-transitive dihedrants among regular $\mathbb{Z}_p$-covers, p a prime, of the above basic graphs needs to be proved (see the argument in the last few paragraphs of the proof of [31, Theorem 2.1] and Lemma 2.6). In Section 6 we show that the only 2-arc-transitive dihedrants among cyclic covers of the graphs $K_{n,n} - nK_2$ are the graphs K_{q+1}^{2d} with $n = q+1$, where q is an odd prime power, and d is a divisor of $\frac{q-1}{2}$ or $q-1$ depending on whether $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$, respectively.

The proof of Theorem 1.2 then follows using Theorem 1.1 and Sections 3, 4, 5, 6.

2. Preliminaries

2.1. Notation and terminology

Throughout this paper groups are finite, and graphs are finite, simple, undirected, and unless specified otherwise, connected. For a graph X we let $V(X)$, $E(X)$ and $A(X)$ denote the vertex set, the edge set, and the arc set of X, respectively. If u and v are adjacent (or neighbors) in X, we denote the corresponding edge by $[u, v]$ (or by its shorter version uv).

For the group-theoretic terminology not defined here we refer the reader to [6,20,43]. By $\mathbb{F}_q$, $\mathbb{Z}_n$ and D_{2n} we denote the finite field of q elements, the cyclic group of order n, and the dihedral group of order $2n$, respectively. A semidirect product of the group N by the group H is denoted by $N : H$. By $V = V(n, q)$, $\mathrm{PG}(V)$, $\mathrm{AG}(V)$, $\mathrm{GL}(n, q)$, $\mathrm{PGL}(n, q)$, $\mathrm{PSL}(n, q)$, and $\mathrm{AGL}(n, q)$, respectively, we denote the n-dimensional linear space of row vectors, projective geometry, affine geometry, general linear group, projective general linear group, projective special linear group, and affine transformation group over $\mathbb{F}_q$. For any $\alpha \in V$ we denote by t_α the translation corresponding to $\alpha \in \mathrm{AG}(V)$, and by $T \cong \mathbb{Z}_p^n$ the translation subgroup of $\mathrm{AGL}(n, p)$. Then $\mathrm{AGL}(n, p) \cong T : \mathrm{GL}(n, p)$. We adopt matrix notation for $\mathrm{GL}(n, p)$, so that $g^{-1} t_\alpha g = (t_\alpha)^g = t_{\alpha g}$ for any $t_\alpha \in T$ and any $g \in \mathrm{GL}(2, p)$.

For a permutation group G on the set Ω, we denote by G_Δ and $G_{\{\Delta\}}$ the pointwise- and setwise-stabilizer of $\Delta \subset \Omega$, respectively. A permutation group G is said to act *semiregularly* on Ω if it has trivial point stabilizers. A transitive and semiregular group is said to be *regular*.

Adopting the terminology of Tutte [41], for $k \geqslant 0$, a *k-arc* in a graph X is a sequence of $k + 1$ vertices $v_1, v_2, \ldots, v_{k+1}$ of X, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. Let X be a graph and G be a subgroup of its automorphism group $\mathrm{Aut}\, X$. We say that X is *(G, k)-arc-transitive* provided G acts transitively on the set of k-arcs of X. In particular, when $G = \mathrm{Aut}\, X$ we say that X is *k-arc-transitive*, and we say that it is *exactly k-arc-transitive* if it is k-arc-transitive but not $(k + 1)$-arc-transitive. Also, 1-arc-transitive graphs and 0-arc-transitive graphs are usually referred to as *arc-transitive* and *vertex-transitive*, respectively.

2.2. Coset graphs and bicoset graphs

Let G be a finite group and H a proper subgroup of G with $\mathrm{Core}(H) = 1$. Let $D = D^{-1}$ be inverse-closed union of some double cosets of H in G. Then the *coset graph* $X = X(G; H, D)$ is defined by taking $V(X) = \{Hg \mid g \in G\}$ as the vertex set and $D(X) = \{(Hg_1, Hg_2) \mid g_2 g_1^{-1} \in D\}$ as the edge-set. It follows that G acts by right multiplication faithfully and vertex-transitively on X. In particular, if $H = 1$ we get a *Cayley graph*. Conversely, each vertex-transitive graph is isomorphic to a coset graph (see [40]).

Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \bigcup_i R d_i L$. By $[G : L]$ and $[G : R]$, we denote the set of cosets G relative to L and R, respectively. Define a bipartite graph $X = \mathcal{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{\{Lg, Rdg\} \mid g \in G, d \in D\}$. This graph is called the *bi-coset graph* of G with respect to L, R and D (see [10]).

Proposition 2.1. *(See [10, Lemmas 2.3, 2.4].)*

(i) *The bicoset graph $X = \mathcal{B}(G, L, R; D)$ is connected if and only if G is generated by elements of $D^{-1}D$.*
(ii) *Let Y be a bipartite graph with bipartition $V(Y) = U(Y) \cup W(Y)$, let G be a subgroup of $\mathrm{Aut}(Y)$ acting transitively on both U and W, let $u \in U(Y)$ and $w \in W(Y)$, and set $D = \{g \in G \mid w^g \in N(u)\}$, where $N(u)$ denotes the neighborhood of u. Then D is a union of double cosets of G_w and G_u in G, and $Y \cong \mathcal{B}(G, G_u, G_w; D)$. In particular, if $uw \in E(Y)$ and G_u acts transitively on its neighbor, then $D = G_w G_u$.*

2.3. Covers of graphs

A graph X is called a *covering* (or *cover*) of a graph Y with the projection $p: X \to Y$ if there is a surjection $p: V(X) \to V(Y)$ such that $p|_{N(x)}: N(x) \to N(y)$ is a bijection for any vertex $y \in V(Y)$ and $x \in p^{-1}(y)$. The graph X is called the *covering graph* and Y is the *base graph*. A covering p is *n-fold* if $|p^{-1}(y)| = n$ for each $y \in V(Y)$. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of X which maps a fibre to a fibre is said to be *fibre-preserving*. The group K of all automorphisms of X which fix each of the fibres setwise is called the *covering transformation group*. A cover X of Y is said to be *regular* (simply, *L-covering*) if there is a subgroup L of K acting freely and transitively, that is, regularly on each fibre. Moreover, if X is connected, then $L = K$.

A purely combinatorial description of a covering was introduced through a voltage graph by Gross and Tucker [19]. Let Y be a graph and K a finite group. A *voltage assignment* (or, *K-voltage assignment*) of the graph Y is a function $f: A(Y) \to K$ with the property that $f(u,v) = f(v,u)^{-1}$ for each $(u,v) \in A(Y)$. The values of f are called *voltages*, and K is called the *voltage group*. The *derived graph* $Y \times_f K$ *from a voltage assignment* f has as its vertex set $V(Y) \times K$ and as its edge set $E(Y) \times K$, so that an edge (e,g) of $Y \times_f K$ joins a vertex (u,g) to $(v, f(u,v)g)$ for $(u,v) \in A(Y)$ and $g \in K$, where $e = uv$. Clearly, the graph $Y \times_f K$ is a covering of the graph Y with the first coordinate projection $p: Y \times_f K \to Y$, which is called the *natural projection*. For each $u \in V(Y)$, $\{(u,g) \mid g \in K\}$ is a fibre of u. Moreover, by defining $(u,g')^g := (u, g'g)$ for any $g \in K$ and $(u,g') \in V(Y \times_f K)$, K can be identified with a subgroup of $\mathrm{Aut}(Y \times_f K)$ fixing each fibre setwise and acting regularly on each fibre. Therefore, p can be viewed as a *K-covering*. Conversely, each connected regular cover X of Y with the covering transformation group K can be described by a derived graph $Y \times_f K$ from some voltage assignment f. Given a spanning tree T of the graph Y, a voltage assignment f is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [18] showed that every regular cover X of a graph Y can be derived from a T-reduced voltage assignment f with respect to an arbitrary fixed spanning tree T of Y. Moreover, the voltage assignment f naturally extends to walks in Y. For any walk W of Y, let f_W denote the voltage of W. Finally, we say that an automorphism α of Y *lifts* to an automorphism $\bar{\alpha}$ of X if $\alpha p = p\bar{\alpha}$, where p is the covering projection from X to Y.

The next two propositions deal with lifting of automorphisms in graph covers. The next two propositions provide information about the relationship between automorphisms of graph covers and their base graphs. The first one is taken from [29, Corollary 4.3], whereas the second one is taken from [8, Proposition 2.2], but it may also be deduced from [30, Corollaries 9.4, 9.7, 9.8].

Proposition 2.2. *Let $X = Y \times_f K$ be a connected regular cover of a graph Y derived from a voltage assignment f with the voltage group K. Then, an automorphism α of Y can be lifted to an automorphism of X if and only if, for each closed walk W in Y, we have $f(W^\alpha) = 1$ if and only if $f(W) = 1$.*

Proposition 2.3. *Let K be a finite group, let $X = Y \times_f K$ be a connected regular cover of a graph Y derived from a voltage assignment f with the voltage group K, and let the lifts of $\alpha \in \mathrm{Aut}\,Y$ centralize K, considered as the group of covering transformations. Then for any closed walk W in Y, there exists $k \in K$ such that $f_{W^\alpha} = kf(W)k^{-1}$. In particular, if K is abelian, $f(W^\alpha) = f(W)$ for any closed W of Y.*

Lemma 2.4. *Let Y be a connected bipartite graph of girth 4 and diameter 3, with bipartition $U \cup W$. Let $X = Y \times_f K$ be a cover of Y, where K is abelian. Suppose that any three vertices $w_1, w_2, w_3 \in W$ have a common neighbor in U. If the voltages of all 4-cycles of Y belong to a subgroup K_0 of K, then also all the voltages of 6-cycles of Y belong to K_0.*

Proof. Let $C = u_1, w_1, u_2, w_2, u_2, w_3, u_1$ be an arbitrary 6-cycle, where $u_i \in U$ and $w_i \in W$. Let w_1, w_2, w_3 have a common neighbor u. If $u \in \{u_1, u_2, u_3\}$, say $u = u_1$, then we have $f(C) = f(u_1, w_1, u_2, w_2, u_1) + f(u_1, w_2, u_3, w_3, u_1) \in K_0$. If $u \notin \{u_1, u_2, u_3\}$, then we have $f(C) = f(u_1, w_1, u, w_3, u_1) + f(w_1, u_2, w_2, u, w_1) + f(w_2, u_3, w_3, u, w_2) \in K_0$. □

The power set of the edge set of a graph X can be thought of as a vector space over $\mathbb{Z}_2$ by taking symmetric difference as addition. The *cycle space* of X is the subspace of this vector space generated by all simple cycles of X.

The next two lemmas need no proof.

Lemma 2.5. *Let Y be a graph and let $\mathcal{B}$ be a set of cycles of Y spanning the cycle space of Y. If X is a graph cover of Y given by a voltage assignment f for which each $C \in \mathcal{B}$ vanishes, then X is disconnected.*

Lemma 2.6. *Let $X \to Y$ be a regular cyclic covering of connected graphs such that some 2-arc-transitive group $G \leqslant \mathrm{Aut}\, X$ projects along $X \to Y$. Then there exists a regular prime cyclic covering $X' \to Y$ such that some 2-arc-transitive group $G' \leqslant \mathrm{Aut}\, X'$ projects along $X' \to Y$.*

2.4. Some group-theoretic results

The following proposition is extracted from a complete list of doubly transitive groups in [3] and [4, Corollary 8.3].

Proposition 2.7. *Let G be a 3-transitive permutation group of degree at least 5. Then one of the following occurs.*

(i) $\mathrm{soc}(G)$ *is 4-transitive*;
(ii) $\mathrm{soc}(G) = M_{22}$ *is 3-transitive but not 4-transitive*;
(iii) $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{Aut}\,\mathrm{PSL}(2, q)$, *where the projective special linear group* $\mathrm{PSL}(2, q)$ *is the socle* $\mathrm{soc}(G)$ *of G which does not act 3-transitively, and G acts on the projective geometry* $\mathrm{PG}(1, q)$ *in a natural way, having degree $q + 1$, with $q \geqslant 5$ an odd prime power*;
(iv) $G = \mathrm{AGL}(m, 2)$, *with $m \geqslant 3$*;
(v) $G = \mathbb{Z}_2^4 : A_7 < \mathrm{AGL}(4, 2)$.

The next two propositions deal with two basic group-theoretic results.

Proposition 2.8. *(See [20, Satz 4.5].) Let H be a subgroup of a group G. Then $C_G(H)$ is a normal subgroup of $N_G(H)$ and the quotient $N_G(H)/C_G(H)$ is isomorphic with a subgroup of* $\mathrm{Aut}\, H$.

Proposition 2.9. *(See [20, Satz 17.5].) Let G be a finite group. Let A and B be two subgroups of G such that A is abelian normal in G, $A \leqslant B \leqslant G$ and $(|A|, |G : B|) = 1$. If A has a complement in B, then A has a complement in G.*

For a group G, we let G' denote the commutator subgroup of G. Recall that a group G is an *extension* of N by H if G has a normal subgroup N such that the quotient group G/N is isomorphic to H. In particular, G is a *proper central extension* of N by H if $N \leqslant Z(G) \cap G'$ is a central subgroup. Such central subgroups are all quotients of a largest group, called the *Schur multiplier* $\mathrm{Mult}(G)$ of G.

Proposition 2.10. *(See [5, p. xv].) The Schur multiplier of the simple group* $\mathrm{PSL}(2, q)$ *is* $\mathbb{Z}_2$ *for* $q \neq 9$; *and* $\mathbb{Z}_6$ *for* $q = 9$.

Proposition 2.11. *Let* G *be a proper central extension of* $K \neq 1$ *by* $\mathrm{PSL}(2, 9)$ *and suppose that* $M \leqslant G$ *and* $M \cong \mathbb{Z}_3^2 : \mathbb{Z}_4$. *Then* $M \cap K \neq 1$.

Proof. Since the Schur multiplier of the simple group $\mathrm{PSL}(2, 9)$ is $\mathbb{Z}_6$, by Proposition 2.10, we have that K is one of $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_6$.

Suppose that $K = \mathbb{Z}_2$. Then $G \cong \mathrm{SL}(2, 9)$ and so G has the unique central involution, say e, while $K = \langle e \rangle$. Since $M \cong \mathbb{Z}_3^2 : \mathbb{Z}_4$, we get $K \leqslant M$.

Suppose that $K = \mathbb{Z}_3$. Assume that $K \nleqslant M$. Take $N \cong \mathbb{Z}_3^2$ be a Sylow 3-subgroup of M. Then $(|K|, |G : NK|) = (3, 80) = 1$. By Proposition 2.9, K has a complement in G, say $W \cong \mathrm{PSL}(2, 9)$, which means $G = K \times W$. Therefore K cannot be contained in G', and then G is not a proper central extension of K by $\mathrm{PSL}(2, 9)$. Therefore, $K \leqslant M$.

Suppose that $K = \mathbb{Z}_6$. Let $K = \langle a \rangle$ and $K_1 = \langle a^2 \rangle$. Then $|K_1| = 3$. Take $N \cong \mathbb{Z}_3^2$ be a Sylow 3-subgroup of M again. Then $(|K_1|, |G : NK_1|) = (3, 160) = 1$. By Proposition 2.9, K_1 has a complement in G, say U, that is $G = K_1 \times U$. Then K_1 cannot be contained in G', a contradiction again. □

3. Covers of the graphs associated with H_{11}

In this section, we shall prove that there are no 2-arc-transitive dihedrants arising as regular cyclic covers of the incidence and nonincidence graphs $B(H_{11})$ or $B'(H_{11})$ of the Hadamard design H_{11}.

Theorem 3.1. *There are no* 2*-arc-transitive dihedrants arising as regular cyclic covers of the graphs* $B(H_{11})$ *or* $B'(H_{11})$.

Proof. In view of Lemma 2.6 and Theorem 1.1 it suffices to show that no such regular $\mathbb{Z}_p$-covers, p a prime, exist. Namely, the existence of a regular cyclic cover of any basic 2-arc-transitive dihedrant implies (using an argument based on minimality of blocks of imprimitivity of the full automorphism group; see the last few paragraphs of the proof of [31, Theorem 2.1]) the existence of such a graph among regular $\mathbb{Z}_p$-covers, p a prime. In fact, we prove here that no 2-arc-transitive connected graphs (at all) can arise as regular $\mathbb{Z}_p$-covers of the graphs $B(H_{11})$ or $B'(H_{11})$.

Let us denote the two bipartition sets of the base graph $Y = B(H_{11})$ (the argument for the base graph $B'(H_{11})$ is analogous and we omit it) by U and W, and let $G \cong \mathrm{PSL}(2, 11)$ be the setwise stabilizer of these two sets. Of course, G is an index 2 subgroup in $\mathrm{Aut}\, Y$. Assume that there exists a 2-arc-transitive regular $\mathbb{Z}_p$-cover X of Y for some prime p. Then the whole group G has to lift, denote it by $\widetilde{G}$. Further, let K be the corresponding group of covering transformations. Then $\widetilde{G}/K = G$. Let $\widetilde{U}$ and $\widetilde{W}$ be the respective bipartition sets of the cover X.

Since $C_{\widetilde{G}}(K)/K$ is normal in $N_{\widetilde{G}}(K)/K = \widetilde{G}/K = G$, a nonabelian simple group, we have that either $C_{\widetilde{G}}(K)/K = \widetilde{G}/K$ or $C_{\widetilde{G}}(K)/K = 1$, that is, $C_{\widetilde{G}}(K)$ is either $\widetilde{G}$ or K. If the latter happens, then in view of Proposition 2.8, we have that $G = \widetilde{G}/K = \widetilde{G}/C_{\widetilde{G}}(K) \leqslant \mathrm{Aut}(K) \cong \mathbb{Z}_{p-1}$, contradicting the simplicity of G. Therefore we must have $C_{\widetilde{G}}(K) = \widetilde{G}$. In particular, $\widetilde{G}$ is a central extension of K by G. Moreover, from $\widetilde{G}'K/K = (\widetilde{G}/K)' = G' = G = \widetilde{G}/K$ we deduce that $\widetilde{G} = \widetilde{G}'K$. Since $K \cong \mathbb{Z}_p$, it follows that either $K \leqslant \widetilde{G}'$ or $K \cap \widetilde{G}' = 1$. We deal with these two possibilities separately.

Case 1. $K \leqslant \widetilde{G}'$.

Since the Schur multiplier of G is $\mathbb{Z}_2$ by Proposition 2.10, it follows that $K \cong \mathbb{Z}_2$ and so $\widetilde{G} \cong \mathrm{SL}(2, 11)$. Since $\mathrm{SL}(2, 11)$ has a unique involution, it does not contain a subgroup isomorphic to A_5, contradicting the fact that vertex stabilizers of $\widetilde{G}$ are isomorphic to A_5, and so this case cannot occur.

Case 2. $K \nleqslant \widetilde{G}'$.

Then $\widetilde{G} = \widetilde{G}' \times K$ where $\widetilde{G}' \cong G \cong \mathrm{PSL}(2, 11)$. For simplicity reasons we replace the symbol $\widetilde{G}'$ with G for the rest of the proof.

Suppose first that G is transitive on one of the bipartition sets, say $\widetilde{U}$. Take a fiber $\widetilde{B}$ inside $\widetilde{U}$ and let $\tilde{u} \in \widetilde{B}$. Note that the vertex stabilizer $G_{\tilde{u}}$ is a subgroup of $G_{\{\widetilde{B}\}} \cong A_5$. Since A_5 does not contain subgroups of index 2 or 3, we must have that $p = 5$, forcing $G_{\tilde{u}} \cong A_4$. Since $N_G(G_{\tilde{u}}) = G_{\tilde{u}}$, it follows that $\tilde{u}$ is the only fixed vertex of $G_{\tilde{u}}$ in $\widetilde{U}$. On the other hand, since $\widetilde{G}_{\{B\}} = G_{\tilde{u}} \times K$, we have that $G_{\tilde{u}} \leqslant \widetilde{G}_{\{B\}}$ centralizes K. Since K is transitive on $\widetilde{B}$, this forces $G_{\tilde{u}}$ to fix all five vertices in $\widetilde{B}$, a contradiction.

Suppose now that G is intransitive on both $\widetilde{U}$ and $\widetilde{W}$. Then G has p orbits of size 11 on both $\widetilde{U}$ and $\widetilde{W}$. But in particular it follows that for any $\tilde{u} \in V(X)$, vertex stabilizers $G_{\tilde{u}}$ and $\widetilde{G}_{\tilde{u}}$ coincide. Hence $\widetilde{G}_{\tilde{u}} = G_{\tilde{u}} \leqslant G$. Let $e = \tilde{u}\tilde{w}$ be an arbitrary edge of X. Then by Proposition 2.1, part (ii), X is isomorphic to a bicoset graph $\mathcal{B}(\widetilde{G}, \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}, D)$, where $D = G_{\tilde{u}}G_{\tilde{w}}$. Since $\widetilde{G}_{\tilde{u}}$ and $\widetilde{G}_{\tilde{w}}$ are contained in G and hence $\langle D^{-1}D\rangle = \langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}\rangle \neq \widetilde{G}$. By Proposition 2.1, part (i), we have that X is disconnected, a contradiction. □

4. Covers of $K_{n,n}$

In this section we show that no 2-arc-transitive dihedrants arise as regular cyclic covers of complete bipartite graphs $K_{n,n}$. We start with a group-theoretic lemma.

Lemma 4.1. *Let n be a positive integer and p a prime. Then the following statements hold.*

- (i) $\mathrm{GL}(n, p)$ *contains an element of order* p^{n-1} *if and only if* $n \in \{1, 2\}$, *where p is arbitrary, or* $(n, p) = (3, 2)$.
- (ii) $\mathrm{AGL}(n, p)$ *contains an element of order* p^n *if and only if* $n = 1$, *in which case* $\mathrm{AGL}(n, p) = \mathbb{Z}_p : \mathbb{Z}_{p-1}$ *with p arbitrary, or else* $(n, p) = (2, 2)$ *and* $\mathrm{AGL}(2, 2) \cong S_4$.

Proof. The lemma clearly holds for $n = 1$. We therefore assume that $n \geqslant 2$.

To prove (i), let $N = \{L = [a_{ij}] \mid a_{ij} = 0 \text{ for } j < i\}$ be the set of strictly upper triangular matrices. These are clearly nilpotent, with $L^n = 0$. Let

$$N_r = \{I + L^r \mid L \in N\}.$$

By [20, Satz 7.1], the set $N_1 = P$ is the Sylow p-subgroup of $\mathrm{GL}(n, p)$, and by [20, Satz 16.5], the set $N_{p^k} = P^{p^k}$ is the set of all p^k-powers of the elements of P. Hence P contains an element of order p^k if and only if $N_{p^k} = \{I\}$ and $N_{p^{k-1}} \neq \{I\}$. This condition is equivalent to $p^{k-1} < n \leqslant p^k$, in view of the fact that $N_r = \{I\}$ if and only if $r \geqslant n$. Since $p^{n-1} \geqslant n$ it follows that the order of an element from P is less than or at most equal to p^{n-1}. Suppose that an element of order p^{n-1} exists. Noting that we have assumed $n \geqslant 2$, the inequality $p^{n-2} < n \leqslant p^{n-1}$ implies either $n = 2$ or $(n, p) = (3, 2)$, proving (i).

In order to prove (ii), first note that for an element $t_v A \in \mathrm{AGL}(n, p)$, where v is a row vector in $V(n, p)$ and A a matrix in $\mathrm{GL}(n, p)$, we have $(t_v A)^r = t_{v(I+A+\cdots+A^{r-1})} A^r$. Next, observe that the homomorphism $\mathrm{AGL}(n, p) \to \mathrm{GL}(n, p)$, defined by the rule $t_v A \mapsto A$, lowers the order of an element of $\mathrm{AGL}(n, p)$ by a factor of at most p. Hence if $t_v A$ is an element of order p^n, then A is of order p^{n-1}. By the first part we have $n = 2$ or $(n, p) = (3, 2)$.

Suppose that $n = 2$. Then $A = I + L$, $L \in N$, is an element of order p in $\mathrm{GL}(2, p)$. As $L^2 = 0$ we have $I + A + A^2 + \cdots + A^{p-1} = \frac{1}{2} p(p-1)L$. Therefore if $p \neq 2$, then $I + A + A^2 + \cdots + A^{p-1} = 0$, implying that $t_v A$ has order p, and not p^2. If $p = 2$, however, we have $I + A = L \neq 0$, and one can choose v in such a way that $vL \neq 0$. But then $I + A + A^2 + A^3 = 6L = 0$, and $t_v A$ indeed has order 4. In this case we have $\mathrm{AGL}(2, 2) \cong S_4$.

Suppose next that $(n, p) = (3, 2)$. If $A = I + L$, $L \in N$, is any matrix of order 4, then $I + A + A^2 + A^3 = I + (I + L) + (I + L^2) + (I + L + L^2) = 0$. Thus, $t_v A$ is of order 4, and hence no element of order 8 exists, completing the proof of Lemma 4.1. □

Theorem 4.2. *There are no 2-arc-transitive connected dihedrants arising as regular cyclic covers of a complete bipartite graph $K_{n,n}$, $n \geqslant 3$.*

Proof. In view of Lemma 2.6 and Theorem 1.1 it suffices to consider regular $\mathbb{Z}_p$-covers, where p is a prime, of $K_{n,n}$. Denote the two bipartition sets of the base graph $Y = K_{n,n}$ by U and W. Let $A \leqslant \mathrm{Aut}\, Y$ be a 2-arc-transitive group of automorphisms of Y, and let $G \leqslant A$ be the corresponding index 2 subgroup of A fixing U and W setwise. Let G_U and G_W be the respective pointwise stabilizers of the action of G on U and W. Clearly, $G_U \cong G_W$. Further, suppose that $X = Y \times_f \mathbb{Z}_p$ is a connected regular $\mathbb{Z}_p$-cover of Y, which is a 2-arc-transitive dihedrants, and such that A lifts. Denote by $\widetilde{A}$ and $\widetilde{G}$ the respective lifts of A and G.

Let us start by identifying certain symmetry properties of Y. First observe that, for each vertex $u \in U$, the stabilizer $A_u = G_u$ is 2-transitive on W, and therefore also $G_u G_U / G_U$ is 2-transitive on W. Consequently, G/G_U acts 2-transitively on W. Similarly, G/G_W acts 2-transitively on U. Recall that each 2-transitive group has a unique minimal normal subgroup coinciding with its socle, and is therefore either nonabelian simple or elementary abelian [6, Theorem 7.2E]. Further, since $G_U \cong (G_U \times G_W)/G_W$, and the latter is a normal subgroup of G/G_W, we have that G_U contains a normal subgroup $T_U \cong \mathrm{soc}(G/G_W)$. Analogously, G_W contains a normal subgroup $T_W \cong \mathrm{soc}(G/G_U)$. Clearly, $T_U \cong T_W$. In what follows we distinguish two cases, depending on whether T_U is simple or elementary abelian.

Case 1. T_U is a nonabelian simple group.

In this case T_U and T_W act 2-transitively on W and U, respectively. Let us consider the group $T = T_U \times T_W$. Note that, as T_U and T_W are nonabelian and simple, the only normal nontrivial proper subgroups of T are T_U and T_W. Therefore each nontrivial quotient of T must be nonabelian.

Clearly, the group T has a lift along the covering projection to, say, the group $\widetilde{T} \leqslant \operatorname{Aut} X$. Then $\widetilde{T}/K \cong T$. Note that, since K is abelian, the centralizer $C_{\widetilde{T}}(K)$, a normal subgroup of $\widetilde{T}$, contains K. Now, by Proposition 2.8, in view of

$$(\widetilde{T}/K)/\bigl(C_{\widetilde{T}}(K)/K\bigr) \cong \widetilde{T}/C_{\widetilde{T}}(K) = N_{\widetilde{T}}(K)/C_{\widetilde{T}}(K) \leqslant \operatorname{Aut} K \cong \mathbb{Z}_{p-1},$$

we deduce that $\widetilde{T}$ has an abelian, and hence trivial quotient. Consequently, $\widetilde{T} = C_{\widetilde{T}}(K)$. In other words, the group T lifts to a group centralizing K. By Proposition 2.3, the voltages of closed walks of Y are constant along each orbit of T in its action on the set of closed walks of Y. In particular, since T acts transitively on the set of all 4-cycles in Y, all of these 4-cycles have the same voltage. Now take three vertices $u_0, u_1, u_2 \in U$ and two vertices $w_0, w_1 \in W$ and consider the corresponding three 4-cycles in the subgraph induced by $\{u_0, u_1, u_2, w_0, w_1\}$,

$$C_0 = u_0 w_0 u_1 w_1 u_0, \qquad C_1 = u_0 w_0 u_2 w_1 u_0, \qquad C_2 = u_1 w_1 u_2 w_0 u_1.$$

By computation, $f(C_2) = f(C_0) - f(C_1) = 0$. Consequently, each 4-cycle in Y has trivial voltage. Since 4-cycles span the cycle space of $K_{n,n}$, Lemma 2.5 implies that the graph X is disconnected, a contradiction.

Case 2. T_U is elementary abelian.

In this case, G/G_W is a subgroup of $\mathrm{AGL}(t, r)$, where r is a prime and $n = r^t$ for some t. Noting that in our case G/G_W contains an element of order r^t, we have by Lemma 4.1 that either $n = r$ or $n = 4$. In what follows we deal with these two cases separately.

Subcase 2.1. $n = r$.

Since G_U and G_W are 2-transitive on W and U, respectively, they both contain a subgroup isomorphic to $\mathbb{Z}_r : \mathbb{Z}_{r-1}$. But then G must be isomorphic to $(\mathbb{Z}_r \times \mathbb{Z}_r) : \mathbb{Z}_{r-1}$. We now distinguish two possibilities, depending on whether $r = p$ or $r \neq p$.

Suppose first that $r = p$. Let $\widetilde{G}$ be a lift of G. Then $\widetilde{G}/K = G$. Let P be a Sylow-p-subgroup of $\widetilde{G}$. Then P is a group of order p^3 and of degree p^2 acting transitively on both $\widetilde{U}$ and $\widetilde{W}$, and for any $\tilde{u} \in E(X)$, $P_{\tilde{u}}$ acts transitively on its neighbors. By Proposition 2.1, X is isomorphic to a bicoset graph $\mathcal{B}(P; P_{\tilde{u}}, P_{\tilde{w}}, D)$, where $\tilde{u}\tilde{w} \in E(X)$ and $D = P_{\tilde{w}} P_{\tilde{u}}$ such that $\langle D^{-1} D\rangle = \langle P_{\tilde{u}}, P_{\tilde{w}}\rangle = P$. Clearly, P is nonabelian. Furthermore, since $\widetilde{G}$ (and so P) contains an element of order p^2, the exponent of P is p^2. Therefore, we may let $P = \langle a, b\rangle$, where $|a| = p^2$ and $|b| = p$, and $a^b = a^{1+p}$. Observe that P has $p + 1$ subgroups of order p, one of which is the center $\langle a^p\rangle$, and the other p subgroups are $\langle ba^{pi}\rangle$, $i \in \mathbb{Z}_p$, which are conjugate in P. For $\tilde{u}\tilde{w} \in E(X)$, we have to assume that $P_{\tilde{u}} = \langle ba^{pi}\rangle$ and $P_{\tilde{w}} = \langle ba^{pj}\rangle$, for $i, j \in \mathbb{Z}_p$. But $\langle ba^{pi}, ba^{pj}\rangle \leqslant \langle b, a^p\rangle \neq P$, a contradiction with Proposition 2.1.

Suppose now that $r \neq p$. Let $\widetilde{G}$ be the lift of G and let R be a Sylow-r-subgroup of $\widetilde{G}$, where $|R| = r^2$. Assume first that $R \leqslant C_{\widetilde{G}}(K)$. Then the abelian group $M := K \times R$ acts transitively on both $\widetilde{U}$ and $\widetilde{W}$, with the vertex stabilizers of order r, acting transitively on the corresponding neighbors' sets. By Proposition 2.1 again, X is isomorphic to a bicoset graph $\mathcal{B}(M; M_{\tilde{u}}, M_{\tilde{w}}, D)$, where $\tilde{u}\tilde{w} \in E(X)$ and $D = M_{\tilde{w}} M_{\tilde{u}}$ such that $\langle D^{-1} D\rangle = \langle M_{\tilde{u}}, M_{\tilde{w}}\rangle = P$. However, $\langle M_{\tilde{u}}, M_{\tilde{w}}\rangle = \langle (K \times R)_{\tilde{u}}, (K \times R)_{\tilde{w}}\rangle = \langle R_{\tilde{u}}, R_{\tilde{w}}\rangle \leqslant R$, a proper subgroup of M, a contradiction.

Assume now that $R \nleqslant C_{\widetilde{G}}(K)$. Then KR is nonabelian and R is not normal in the group KR (of order pr^2). Applying the Sylow Theorem for the prime r to the group KR, it follows that r divides $p-1$. In particular, $(p, r-1) = 1$. By Proposition 2.9, it follows that K has a complement S in $\widetilde{G}$, isomorphic to G. Then $\widetilde{G} = K : S$. Now we may assume that $S = (\langle a\rangle \times \langle b\rangle) : \langle c\rangle \cong (\mathbb{Z}_r \times \mathbb{Z}_r) : \mathbb{Z}_{r-1}$, where $R = \langle a\rangle \times \langle b\rangle$ and both $\langle a\rangle : \langle c\rangle$ and $\langle b\rangle : \langle c\rangle$ are Frobenius groups. Consider the conjugacy action of $\langle a\rangle : \langle c\rangle$ on K. Since $\operatorname{Aut} K \cong \mathbb{Z}_{p-1}$, an abelian group, the kernel of this action must be nontrivial. Since $\langle a\rangle$ is the unique nontrivial proper normal subgroup of $\langle a\rangle : \langle c\rangle$, it follows that $\langle a\rangle$ is contained in the kernel, that is $a \in C_{\widetilde{G}}(K)$. Similarly, $b \in C_{\widetilde{G}}(K)$. Therefore, $R \leqslant C_{\widetilde{G}}(K)$, a contradiction.

Subcase 2.2. $n = 4$.

Set $U = \{1, 2, 3, 4\}$ and $W = \{1', 2', 3', 4'\}$. Since A_4 and S_4 are the only 2-transitive subgroups of S_4 and since $G \leqslant A$ contains an element of order 4 acting transitively on both U and W, we have that the induced action of G on both U and W is isomorphic with S_4. Therefore, G contains a subgroup $T = (B_1 \times B_2) : D_6$, where B_1 and B_2 are the Klein 4-groups on U and W, respectively, and D_6 acts faithfully on both U and W. Now T has a lift, say $\widetilde{T}$, so that $\widetilde{T}/K = T$. Then using Proposition 2.8 we get that $C_{\widetilde{T}}(K)/K$ must contain a subgroup isomorphic to $B_1 \times B_2$. In other words, the lift of $B_1 \times B_2$ is a subgroup of $C_{\widetilde{T}}(K)$. Take an arbitrary 4-cycle C in Y, say $C = 1, 1', 2, 2', 1'$, and let $g = (1'2')(3'4') \in B_2$. Then by Proposition 2.3, we have $f(C) = f(C^g) = -f(C)$, which implies that $p = 2$. However, a direct checking shows that $K_{4,4}$ does not have any 2-fold connected 2-arc-transitive covers. $\square$

5. Covers of projective geometry graphs

In this section, let $n \geqslant 3$ and $q = p^l$ for a prime p. Let $\mathrm{V}(n,q)$, $\mathrm{PG}(n-1,q)$ be defined as in Section 2.1, while $P(\Delta)$ denotes the subspace in $\mathrm{PG}(n-1,q)$ corresponding to a subspace Δ in $\mathrm{V}(n,q)$. Denote by U and W, respectively, the sets of points and hyperplanes of $\mathrm{PG}(n-1,q)$. Finally, let $B(\mathrm{PG}(n-1,q))$ and $B'(\mathrm{PG}(n-1,q)$ denote the corresponding (point-hyperplane) incidence graph and the (point-hyperplane) nonincidence graph, respectively.

We start by giving a lemma identifying certain properties of $\mathrm{PG}(n-1,q)$ as well as of the graphs $B(\mathrm{PG}(n-1,q))$.

Lemma 5.1. *Let $Y = B(\mathrm{PG}(n-1,q))$. Then the following hold.*

(i) *The diameter of Y is 3, and furthermore, Y has 4-cycles if and only if $n \geqslant 4$.*
(ii) *For $n \geqslant 4$, any three points are contained in some hyperplanes, and dually any three hyperplanes intersect at some points.*
(iii) *For $n = 3$, $\mathrm{PSL}(3,q)$ acts transitively on 6-cycles of Y while fixing U and W setwise.*
(iv) *For $n \geqslant 4$, $\mathrm{PSL}(n,q)$ acts transitively on 4-cycles of Y while fixing U and W setwise.*

Proof. Part (i) follows directly from the fact that in $\mathrm{PG}(n-1,q)$ any two points are contained in $\frac{q^{n-2}-1}{q-1}$ hyperplanes and any two hyperplanes intersect at $\frac{q^{n-2}-1}{q-1}$ points.

Part (ii) is clear.

As for part (iii), let $T = \mathrm{PSL}(3,q)$. It is well known that T acts 2-transitively on both points and lines of $\mathrm{PG}(2,q)$. We may therefore fix two lines, say

$$w_1 = \langle \langle v \rangle \mid v = (x, y, 0) \in \mathrm{V} \setminus \{0\} \rangle,$$
$$w_2 = \langle \langle v \rangle \mid v = (x, 0, z) \in \mathrm{V} \setminus \{0\} \rangle,$$

and consider 6-cycles containing these two lines. Then the elements of the subgroup $T_{w_1,w_2} = T_{w_1} \cap T_{w_2}$ are of the form $\bar{g} = gZ(\mathrm{GL}(3, q))$, where

$$g = \begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ e & 0 & c \end{pmatrix},$$

and where $abc = 1$. Clearly T_{w_1,w_2} fixes the 2-arc $w_1, w_1 \cap w_2, w_2$. Moreover, the 6-cycles containing this 2-arc are of the form: $C = w_1, w_1 \cap w_2, w_2, u_2, \langle u_1, u_2 \rangle, u_1, w_1$, where for $i = 1, 2$, $u_i \in w_i \setminus w_1 \cap w_2$. Set

$$Q_1 = \langle \bar{g} \mid \bar{g} \in T,\ a = b = c = 1,\ e = 0 \rangle,$$
$$Q_2 = \langle \bar{g} \mid \bar{g} \in T,\ a = b = c = 1,\ d = 0 \rangle.$$

Then for each i, we have that Q_i acts transitively on $w_i \setminus w_1 \cap w_2$ and fixes w_j pointwise for $j = \{1, 2\} \setminus \{i\}$, which implies that $Q_1 \times Q_2$ acts transitively on the set of 6-cycles containing the 2-arc $w_1, w_1 \cap w_2, w_1$. Consequently, T acts transitively on 6-cycles of Y.

Finally, to prove part (iv) let $T = \mathrm{PSL}(n, q)$, $n \geqslant 4$. Again, as in part (iii), since T acts 2-transitively on the sets of points and hyperplanes, we may fix two hyperplanes, say

$$w_1 = \langle \langle v \rangle \mid v = (0, x_2, x_3, \ldots, x_n) \in \mathrm{V} \setminus \{0\} \rangle,$$
$$w_2 = \langle \langle v \rangle \mid v = (x_1, 0, x_3, \ldots, x_n) \in \mathrm{V} \setminus \{0\} \rangle.$$

Then

$$w_1 \cap w_2 = \langle \langle v \rangle \mid v = (0, 0, x_3, \ldots, x_n) \in \mathrm{V} \setminus \{0\} \rangle.$$

The elements of the subgroup T_{w_1,w_2} are of the form $\bar{g} = gZ(\mathrm{GL}(n, q))$, where

$$g = \begin{pmatrix} a_{11} & 0 & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & & & \\ \cdots & & & A & \\ 0 & 0 & & & \end{pmatrix},$$

and where $A \cong \mathrm{GL}(n-2, q)$ and $a_{11}a_{22}|A| = 1$. Let H be the subgroup of T, whose elements satisfy $a_{1j} = a_{2j} = 0$ for $j \neq 1, 2$. (In other words, all the entries on top of A in the above expression for g are 0.) Then H acts 2-transitively on $w_1 \cap w_2$, which implies that T acts transitively on the set of 4-cycles containing w_1 and w_2. Therefore, T acts transitively on the set of all 4-cycles in Y. □

We are now ready to prove the main result of this section.

Theorem 5.2. *Let $n \geqslant 3$ and let Y be either $B(\mathrm{PG}(n-1, q))$ or $B'(\mathrm{PG}(n-1, q)$. Then there are no connected 2-arc-transitive dihedrants, arising as regular cyclic covers of Y.*

Proof. In view of Lemma 2.6 and Theorem 1.1 it suffices to consider regular $\mathbb{Z}_p$-covers, p a prime. We distinguish two cases depending on the base graph.

Case 1. $Y = B(\mathrm{PG}(n-1, q)$.

Suppose that $X := Y \times_f K$, where $K \cong \mathbb{Z}_p$, is a connected regular $\mathbb{Z}_p$-cover of Y such that a 2-arc-transitive subgroup $A \leqslant \mathrm{Aut}\,Y$ lifts. Let $G \leqslant A$ be the corresponding index 2 subgroup of A fixing the two sets of bipartition U (of points) and W (of hyperplanes) setwise. Denote by $\widetilde{A}$ and $\widetilde{G}$ the lifts of A and G, respectively. Then $G \cong \mathrm{PSL}(n, q)$. Further, $\widetilde{G}/K \cong G$ and using Proposition 2.8 we may show (as in previous sections) that $\widetilde{G} = C_K(\widetilde{G})$. In what follows we divide the proof into two cases depending on whether $n = 3$ or $n \geqslant 4$.

Subcase 1.1. $n = 3$.

By part (i) of Lemma 5.1, Y does not contain 4-cycles, but it contains 6-cycles.

Any point $u \in U$ is contained in at least three lines, say $w_i \in W$, $i = 1, 2, 3$, that is, $u = \bigcap_{i=1}^{3} w_i$. Take a line w not containing u and consider the following 6-cycles C_{jk}, for $j, k \in \{1, 2, 3\}$ distinct:

$$C_{jk} = u, w_j, w_j \cap w, w, w \cap w_k, w_k, u.$$

Since, by part (iii) of Lemma 5.1, G is transitive on 6-cycles in Y, it follows, combining Proposition 2.3 and the fact that $\widetilde{G} = C_K(\widetilde{G})$, that all 6-cycles have the same voltages. Now $f(C_{12}) = f(C_{13}) + f(C_{32})$, and consequently $f(C_{jk}) = 0$. Hence all 6-cycles have trivial voltages. But then, since the cycle space of Y is spanned by 6-cycles, Lemma 2.5 forces X to be disconnected, a contradiction.

Subcase 1.2. $n \geqslant 4$.

In this case Y contains both 4-cycles and 6-cycles by part (i) of Lemma 5.1. For any two hyperplanes $w_1, w_2 \in W$ we have $|w_1 \cap w_2| = \frac{q^{n-2}-1}{q-1} \geqslant 3$. Since G acts transitively on the set of 4-cycles (while fixing the sets U and W of points and hyperplanes setwise) by part (iv) of Lemma 5.1, it follows that all 4-cycles have the same voltages. (As above we use the fact $\widetilde{G} = C_K(\widetilde{G})$ and Proposition 2.3.) Choosing arbitrary three points $u_i \in w_1 \cap w_2$, where $i = 1, 2, 3$, and considering the 4-cycles: $C_{jk} = w_1, u_j, w_2, u_k, w_1$, where $j, k \in \{1, 2, 3\}$, we have $f(C_{12}) = f(C_{13}) + f(C_{32})$, which implies $f(C_{jk}) = 0$. Therefore, the voltages of all 4-cycles are trivial. Since any three hyperplanes intersect as some points, it follows from Lemma 2.4 that the voltages on all 6-cycles are trivial too. Since 4-cycles and 6-cycles span the cycle space of Y, Lemma 2.5 implies that the graph X is disconnected, a contradiction.

Case 2. $Y = B'(\mathrm{PG}(n-1, q)$.

Note that both graphs $B(\mathrm{PG}(n-1, q))$ and $B'(\mathrm{PG}(n-1, q))$ have the same automorphism group acting on the vertices. Assume that there is a cover X of Y satisfying our conditions. By $\widetilde{U}$ and $\widetilde{W}$, we denote the two bipartition sets of X corresponding to U and W, respectively. Let A, $\widetilde{A}$, G, $\widetilde{G}$, and $K = \mathbb{Z}_p$ have the same meanings as in Case 1. Then $\widetilde{G}/K \cong G = \mathrm{PSL}(n, q)$.

For any edge $\tilde{u}\tilde{w} \in E(X)$, $\tilde{u} \in \widetilde{U}$, $\tilde{w} \in \widetilde{W}$, it follows from Proposition 2.1 that X is isomorphic to one of bi-coset graphs $\mathcal{B}(\widetilde{G}; \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}, \widetilde{G}_{\tilde{u}}\widetilde{G}_{\tilde{w}})$, where $\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}} \rangle = \widetilde{G}$.

We claim that for any $\tilde{g} \in \widetilde{G}$, we have $\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle = \widetilde{G}$. First by the properties of $G = \mathrm{PSL}(n, q)$ acting on Y, we know that $\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle K/K \cong G$. If $K \leqslant \langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle$, then $\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle = \widetilde{G}$. If $K \nleqslant$

$\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle$, then $\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle \cong G$, and we set $T = \langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{g}} \rangle$ so that $\widetilde{G} = T \times K$. Set $\tilde{g} = tk$, where $t \in T$ and $k \in K$. Then $\widetilde{G}_{\tilde{w}} = (\widetilde{G}_{\tilde{w}}^{\tilde{g}})^{g^{-1}} \leqslant T^{\tilde{g}^{-1}} = T^{k^{-1}t^{-1}} = T$, which forces $T = \widetilde{G}$, a contradiction.

Now take an element $h \in G$ such that $uw^h \in E(B(\mathrm{PG}(n-1,q)))$. Let $\tilde{h}$ be one of the lifts of h. Then we define a graph X_0 with vertex-set $V(X)$ and edge-set $\{(\tilde{u}, \tilde{w}^{\tilde{h}})^{\tilde{g}} \mid \tilde{g} \in \widetilde{G}\}$. Then X_0 is a regular covering of $B(\mathrm{PG}(n-1,q))$. Moreover, using the arguments in the above paragraph, we have that $\langle \widetilde{G}_{\tilde{u}}, \widetilde{G}_{\tilde{w}}^{\tilde{h}} \rangle = \widetilde{G}$, implying that X_0 is connected, contradicting Case 1. This completes the proof of Theorem 5.2. □

6. Covers of $K_{n,n} - nK_2$

In this section we determine all 2-arc-transitive dihedrants arising as regular cyclic covers of the graphs $K_{n,n} - nK_2$.

Let q be an odd prime power and let $\mathbb{F}_q$ and $\mathbb{F}_q^* = \langle \theta \rangle$, as in the introductory section. Set

$$U = \mathrm{PG}(1,q) = \{\infty, 0, 1, \ldots, q-1\},$$
$$W = \{\infty', 0', 1', \ldots, (q-1)'\}.$$

Let $Y = K_{n,n} - nK_2$, where $V(Y) = U \cup W$ and where $ij' \in E(Y)$ for $i, j \in U$ and $i \neq j$. Let d be a divisor of $\frac{q-1}{2}$ when $q \equiv 1 \pmod 4$ and a divisor of $q-1$ when $q \equiv 3 \pmod 4$, respectively. It will be convenient to redefine the regular $\mathbb{Z}_d$-covers $K_{q+1}^{2d} = Y \times_f K$ of the graphs $K_{q+1,q+1} - (q+1)K_2$, introduced in Section 1, using multiplicative notation for the group of covering transformations $K = \langle r \rangle \cong \mathbb{Z}_d$. We set

$$f(\infty, i') = f(i, \infty') = 1, \quad \text{for } i \neq \infty,$$
$$f(i, j') = r^h \quad \text{for } i, j \neq \infty,\ i - j = \theta^h.$$

We may now embark on a long and tedious proof of the fact that the above graphs K_{q+1}^{2d} are the only 2-arc-transitive dihedrants among cyclic covers of complete bipartite graphs minus a matching.

Theorem 6.1. *Let $n \geqslant 3$, and let X be a connected 2-arc-transitive dihedrant arising as a regular cyclic cover of $K_{n,n} - nK_2$. Then $n = q+1$ for some odd prime power $q = p^l$, and $X \cong K_{q+1}^{2d}$ for some d dividing $\frac{q-1}{2}$ if $q \equiv 1 \pmod 4$, and dividing $q-1$ if $q \equiv 3 \pmod 4$.*

Proof. Denote the two sets of bipartition of the base graph $Y = K_{n,n} - nK_2$ by $U = \{1, 2, \ldots, n\}$ and $W = \{1', 2', \ldots, , n'\}$, where $ii' \notin E(Y)$. Let $A \leqslant \mathrm{Aut}\, Y$ be a 2-arc-transitive group of automorphisms of the base graph, and let $G \leqslant A$ be the corresponding index 2 subgroup of A fixing the two sets of bipartition setwise. Then $A = G \times \mathbb{Z}_2$ and G has a faithful 3-transitive representations on both U and W, where $G_u = G_w$ for each pair of nonadjacent vertices $u \in U$ and $w \in W$.

Further, let $K \cong \mathbb{Z}_d$ (written multiplicatively) be the group of covering transformations of the respective cyclic covering projection $X \to Y$. Denote by $\widetilde{A}$ and $\widetilde{G}$ the lifts of A and G, respectively. Because of 2-arc-transitivity, G must be one of 3-transitive groups listed in Proposition 2.7. The analysis depends on whether the group is affine or it has a nonabelian simple socle. It is the latter, in one special case, that will give us the desired graphs.

Suppose first that G is an affine group. Since G must contain a cyclic regular subgroup, it follows from Proposition 2.7 and Lemma 4.1 that $G \cong S_4$, forcing Y to be isomorphic to the cube $Q_3 = K_{4,4} - 4K_2$. However it may be deduced from [33, Theorem 1] as well as from [15, Theorem 1.1] that the Möbius–Kantor graph $\mathrm{GP}(8,3) \cong K_4^4$ is the only 2-arc-transitive dihedrant cyclically covering the cube.

Suppose next that G has a nonabelian simple socle $T = \mathrm{soc}(G)$. Let $\widetilde{T}$ denote the lift of T. Then $\widetilde{T}/K \cong T$. As in the previous sections, a standard argument shows that $K \leqslant Z(\widetilde{T})$. We need to distinguish three different cases.

Case 1. T acts 4-transitively on both U and W.

Then $n \geqslant 6$. Moreover, T fixes U and W setwise and acts transitively on the set of 4-cycles i, h', j, k', i, where $i, j \in U$ and $h', k' \in W$. Combining the fact that $K \leqslant Z(\widetilde{T})$ with Proposition 2.3, we have that the voltages on all 4-cycles are equal. Choose arbitrary five vertices, say with no loss of generality, $0, 1, 2, 3, 4 \in U$ and the corresponding five vertices $0', 1', 2', 3', 4'$ in W. Consider the 4-cycles $C_1 = 0, 1', 3, 2', 0$, $C_2 = 0, 2', 4, 1', 0$, and $C_3 = 1', 3, 2', 4, 1'$. Then $f(C_1)f(C_2) = f(C_3)$, which implies that $f(C_3) = 1$ and so all voltages have to be equal to 1. But then by Lemma 2.4 also all 6-cycles have trivial voltages. Since 4-cycles and 6-cycles generate the cycle space of Y, Lemma 2.5 implies that X is disconnected, a contradiction.

Case 2. $T \cong M_{22}$.

Let $V = V(3,4)$ be a 4-dimensional vector space over $\mathbb{F}_4$. The point stabilizers of the action of T (on the set of size 22), are isomorphic to $H \cong \mathrm{PSL}(3,4)$. We conveniently relabel the sets U and W as $U = \{\infty\} \cup \{\langle v\rangle \mid v \in V\}$ and $W = \{\infty'\} \cup \{\langle v'\rangle \mid v \in V\}$, where $T_\infty = T_{\infty'} = H$ and $T_v = T_{v'}$ for any $v \in V$.

We now show that $d = 2$. Since T acts 3-transitively on U and W, we only need to consider the 4-cycles containing ∞ and $\langle v_0\rangle$ where $v_0 = (1,0,0)$. Now an element $\bar{g}$ in $H_{\langle v_0\rangle}$ is of the form

$$g = \begin{pmatrix} (bc)^{-1} & 0 & 0 \\ d & b & a \\ e & f & c \end{pmatrix},$$

where $a, b, c, d, e, f \in \mathbb{F}_4$ and $bc \neq 0$. We now show that for any two vertices in $U \setminus \{\infty, \langle v_0\rangle\}$, there exists an element in $T_{\infty,\langle v_0\rangle}$ interchanging these two vertices. Since $T_{\infty,\langle v_0\rangle} = H_{\langle v_0\rangle}$ acts transitively on $U \setminus \{\infty, \langle v_0\rangle\}$, it suffices to see that for a fixed vertex, say $\langle v_3\rangle$ where $v_3 = (0,1,0)$, and any vertex $\langle v\rangle \in U \setminus \{\infty, \langle v_0\rangle, \langle v_3\rangle\}$, there exists an element in $T_{\infty,\langle v_0\rangle}$ interchanging $\langle v_3\rangle$ and $\langle v\rangle$. Clearly,

$$H_{\langle v_0\rangle,\langle v_3\rangle} = \langle \bar{g} \in H_{\langle v_0\rangle} \mid a = d = 0\rangle.$$

It may be seen that $H_{\langle v_0\rangle,\langle v_3\rangle}$ has the following two orbits on $\mathrm{PG}(V) \setminus \{\langle v_0\rangle, \langle v_3\rangle\}$:

$$\Delta_1 = \bigl\{\bigl\langle(1, y, 0)\bigr\rangle \bigm| y \neq 0\bigr\},$$
$$\Delta_2 = \bigl\{\bigl\langle(x, y, 1)\bigr\rangle \bigm| x, y \in \mathbb{F}_4\bigr\},$$

where $|\Delta_1| = 3$ and $|\Delta_2| = 16$. Set $v_1 = (1,1,0)$ and $v_2 = (0,0,1)$. Then

$$\bigl\{\langle v_i\rangle^{\bar{g}} \bigm| \bar{g} \in H_{\langle v_0\rangle,\langle v_3\rangle}\bigr\} = \Delta_i,$$

where $i \in \{1, 2\}$. Choose elements $\bar{g}_1, \bar{g}_2 \in H_{\langle v_0\rangle}$, where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then for each $i \in \{1, 2\}$ we have that $\bar{g}_i$ interchanges $\langle v_3\rangle$ and $\langle v_i\rangle$. From the transitivity of $H_{\langle v_0\rangle,\langle v_3\rangle}$ on Δ_i we have that for any vertex $\langle v\rangle$ in Δ_i, there exists an element in $T_{\infty,\langle v_0\rangle}$ interchanging $\langle v_3\rangle$ and $\langle v\rangle$.

It follows that the voltages of the 4-cycle $C = \infty, v_3', v_0, v', \infty$ and its inverse $C^{-1} = \infty, v', v_0, v_3', \infty$ coincide. It follows that $f(C) = f(C)^{-1}$, and so $f(C) = 1$ or $f(C) = r^{\frac{d}{2}}$, where recall that $K = \langle r\rangle$. If $d \neq 2$, then by Lemma 2.4, all the voltages on 6-cycles are in $\langle r^{\frac{d}{2}}\rangle$, which forces X to be disconnected. Hence $d = 2$.

Now consider the 4-cycles of the form

$$C(v) = \infty, \langle v_3'\rangle, \langle v_0\rangle, \langle v'\rangle, \infty,$$

where $\langle v\rangle \in \Delta_1 \cup \Delta_2$. Since $T_{\infty,\langle v_0\rangle,\langle v_3\rangle}$ is transitive on Δ_i, $i \in \{1, 2\}$, all the cycles $C(v)$ for $v \in \Delta_i$, $i \in \{1, 2\}$, have the same voltage.

Let $\mathbb{F}_4^* = \langle b\rangle$, set $\langle v_4\rangle = \langle(1, b, 0)\rangle \in \Delta_1$ and take $\bar{g}' \in T_{\infty,\langle v_0\rangle}$, where

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & b & 0 \\ 0 & 0 & b^2 \end{pmatrix}.$$

Then $|\bar{g}'| = 3$ and $\bar{g}'$ permutes cyclically $\langle v_3\rangle$, $\langle v_4\rangle$ and $\langle v_1\rangle$. Set $C = \infty, \langle v_4'\rangle, \langle v_0\rangle, \langle v_1'\rangle, \infty$. Then $C(v_4)^{\bar{g}'} = C$ and $C^{\bar{g}'} = C(v_1)^{-1}$. Since $d = 2$, we have $f(C(v_4)) = f(C) = f(C(v_1))$. But $f(C(v_1)) = f(C(v_4))f(C)$. Hence $f(C(v_1)) = 1$, and so $f(C(v)) = 1$ for all $\langle v\rangle \in \Delta_1$.

By taking the representative $\langle v_5\rangle = \langle(1, b, 0)\rangle \in \Delta_2$ and $\bar{g}'' \in T_{\infty,\langle v_0\rangle}$, where

$$g'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b^2 \end{pmatrix},$$

a similar argument as above shows that $f(C(v)) = 1$ for all $\langle v\rangle \in \Delta_2$, too. Since M_{22} acts transitively on the set of 2-arcs with the middle vertex in U as well as on those with the middle vertex in W, it follows that the voltages on all the 4-cycles are trivial. By Lemma 2.4 again, also the voltages on all 6-cycles are trivial. Therefore, by Lemma 2.5, the graph X is disconnected, a contradiction.

Case 3. $T = \mathrm{PSL}(2, q)$.

Note that in this case $G = \mathrm{PGL}(2, q)$ has a lift. Since $\widetilde{A}$ contains a regular dihedral subgroup in its action on $V(X)$, it follows that either $\widetilde{G}$ contains a cyclic subgroup, say $\langle a\rangle$, acting regularly on both $\widetilde{U}$ and $\widetilde{W}$, or it contains a dihedral subgroup, say D, acting regularly on both $\widetilde{U}$ and $\widetilde{W}$. But if the latter occurs, then D/K acts regularly on both U and W. However, in its action on $\mathrm{PG}(1, q)$, the group $\mathrm{PGL}(2, q)$ does not contain a regular dihedral subgroup. Consequently, the first possibility occurs. In particular, $\langle a^{(1+q)}\rangle = K$ and $a \in C_{\widetilde{G}}(K)$.

Since $(C_{\widetilde{G}}(K)/K) \lhd (\widetilde{G}/K) \cong \mathrm{PGL}(2,q)$, we know that $C_{\widetilde{G}}(K)/K$ is isomorphic to one of the following: 1, $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$. But since $(\widetilde{G}/K)/(C_{\widetilde{G}}(K)/K) \cong \widetilde{G}/C_{\widetilde{G}}(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$, which is abelian, the first possibility cannot occur. But neither can the second possibility. Namely, $\langle a\rangle \in C_{\widetilde{G}}(K)$ but aK/K cannot be contained in the subgroup of $\widetilde{G}/K$, isomorphic to $\mathrm{PSL}(2,q)$. We must therefore have that $C_{\widetilde{G}}(K) = \widetilde{G}$, and in particular $K \leqslant Z(\widetilde{G})$. Since $\widetilde{G}$ is nonsolvable and $\widetilde{G}/K \cong \mathrm{PGL}(2,q)$, there exists a positive integer k such that $\widetilde{G}^{(k)} = \widetilde{G}^{(k+1)}$, that is, the kth and the $(k+1)$th derived subgroups coincide. Hence $\widetilde{G}^{(k)} \cap K \leqslant \widetilde{G}^{(k)} = (\widetilde{G}^{(k)})'$, and moreover,

$$\widetilde{G}^{(k)}/\bigl(\widetilde{G}^{(k)} \cap K\bigr) \cong \widetilde{G}^{(k)}K/K \cong (\widetilde{G}/K)^{(k)} \cong \bigl(\mathrm{PGL}(2,q)\bigr)^{(k)} \cong \mathrm{PSL}(2,q).$$

By Proposition 2.10, the Schur Multiplier of $\mathrm{PSL}(2,q)$ is $\mathbb{Z}_2$ for $q \neq 9$ and $\mathbb{Z}_6$ for $q = 9$. In what follows, we set $S = \widetilde{G}^{(k)}$ and take $\tilde{u} \in \widetilde{U}$, so that $S/(S \cap K) \cong \mathrm{PSL}(2,q)$ and $S_{\tilde{u}} \cong \mathbb{Z}_p^l : \mathbb{Z}_{\frac{q-1}{2}}$, where $q = p^l$.

Suppose that $q = 9$ and $S \cap K \neq 1$. Since $S_{\tilde{u}} \cong \mathbb{Z}_3^2 : \mathbb{Z}_4$, by Proposition 2.11, $S_{\tilde{u}} \cap K \neq 1$. However, a central element cannot be contained in a vertex-stabilizer. Therefore, if $q = 9$, then $S \cap K = 1$.

In what follows, we need to consider two cases, depending on whether $S \cap K = 1$ or $S \cap K = \mathbb{Z}_2$. Moreover, in the latter case, $S \cong \mathrm{SL}(2,q)$, which has the unique involution e, while $e \in K$. Since $S_{\tilde{u}} \cong \mathbb{Z}_p^l : \mathbb{Z}_{\frac{q-1}{2}}$ and e cannot be contained in $S_{\tilde{u}}$, we get $\frac{q-1}{2}$ is odd, that is $q \equiv 3 \pmod 4$.

Subcase 3.1. $S \cap K = 1$.

We shall first determine the structure of the lifted group $\widetilde{A}$, and then move on to determine the corresponding coset graphs, and finally we shall show that these graphs are isomorphic to the graphs K_{q+1}^{2d}.

We have $S \cong \mathrm{PSL}(2,q)$ and $|\widetilde{G} : S \times K| = 2$. We may identify S with $\mathrm{PSL}(2,q)$. Since $|a| = d(q+1)$ and $|\langle a\rangle \cap S| = \frac{q+1}{2}$ and $|\langle a\rangle \cap K| = d$, we have that $(\frac{q+1}{2}, d) = 1$.

Suppose first that d is even. Then $\frac{q+1}{2}$ is odd, and $z = a^{\frac{q+1}{2}}$ has order $2d$. Suppose next that d is odd. Then $\langle a^d\rangle \cap K = 1$ and $\langle a^d, S\rangle \cong \mathrm{PGL}(2,q)$. Taking s, an involution of $\langle a^d, S\rangle \setminus S$ and setting $z = sa^{(1+q)}$, we have that $|z| = 2d$ again. In both cases, $K = \langle z^2\rangle$ and $\widetilde{G} = S : \langle z\rangle$. Clearly, $(S : \langle z\rangle)/\langle z^2\rangle \cong \mathrm{PGL}(2,q)$.

To determine $\widetilde{A}$, let xK be an involution in $\widetilde{A}/K \cong \mathrm{PGL}(2,q) \times \mathbb{Z}_2$ such that $[xK, gK] = K$ for any $g \in \widetilde{A}$. Then $x^2 \in K$ and x interchanges $\widetilde{U}$ and $\widetilde{W}$.

For any $t \in S$, we have $t^x = tk$ for some $k \in K$. Since S is a characteristic subgroup of $\widetilde{G}$, we have $t^x \in S$. This forces $k = t^{-1}t^x \in S \cap K = 1$ and so $[x, S] = 1$.

Suppose that $z^x = zk$, for $k \in K$. Since z^2 generates K, we have $k = z^{2i}$ for some i, and so $z^x = z^{1+2i}$. Let b be an involution in $\widetilde{A}$ such that $\langle a, b\rangle$ is a regular dihedral subgroup of $\widetilde{A}$. Then for some $t_1 \in S$ and i_1 such that $(i_1, 2d) = 1$, we have $a = t_1 z^{i_1}$. Further there are $t_2 \in S$ and i_2 such that $b = t_2 z^{i_2} x$. Then $a^b = a^{-1}$ implies that

$$\bigl(t_1 z^{i_1}\bigr)^{t_2 z^{i_2} x} = \bigl(t_1 z^{i_1}\bigr)^{-1},$$

that is,

$$1 = \bigl(t_2^{-1} t_1 z^{i_1} t_2\bigr)^{z^{i_2} x} t_1 z^{i_1} = \bigl(t_2^{-1} t_1\bigr)^{z^{i_2} x} \bigl(z^{i_1}\bigr)^x (t_2)^{z^{i_2} x} t_1 z^{i_1} = \bigl(t_2^{-1} t_1\bigr)^{z^{i_2} x} z^{2i_1(1+i)} \bigl((t_2)^{z^{i_2} x} t_1\bigr)^{z^i},$$

which forces $z^{2i_1(1+i)} \in S$, and so $i = -1 \pmod d$. Therefore, we have $z^x = z^{-1}$. Moreover, from $b^2 = 1$, we get

$$1 = t_2 z^{i_2} x t_2 z^{i_2} x = t_2 z^{i_2} x^2 \big(t_2 z^{i_2}\big)^x = x^2 t_2 z^{i_2} (t_2 z^{i_2})^x = x^2 t_2 z^{i_2} t_2 z^{-i_2} = x^2 t_2 t^{z^{-i_2}},$$

which forces $x^2 \in S$. Then $x^2 \in K \cap S = 1$, that is, we have $x^2 = 1$.

Summarizing the above arguments, we obtain the structure of $\widetilde{A}$, that is, $\widetilde{A} = (S : \langle z\rangle) : \langle x\rangle$, where $K = \langle z^2\rangle \leqslant Z(\widetilde{G})$, $|x| = 2$, $[S, x] = 1$ and $z^x = z^{-1}$. Moreover, $(S : \langle z\rangle)/\langle z^2\rangle \cong \mathrm{PGL}(2, q)$. Note that our graph X is isomorphic to some coset graph of the group $\widetilde{A}$. An analysis of coset graphs of $\widetilde{A}$ is therefore in order. (Among others, we shall see that $|K|$ divides $\frac{q-1}{2}$.)

The following notation for particular elements of $\mathrm{PGL}(2, q)$ is introduced for later use:

$$t_i = \overline{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}}, \qquad c = \overline{\begin{pmatrix} 0 & \theta \\ -1 & 0 \end{pmatrix}}, \qquad y = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}},$$

where $i \in \mathbb{F}_q$. Then $yc = \overline{\left(\begin{smallmatrix} 1 & 0 \\ 0 & \theta \end{smallmatrix}\right)}$. Let $Q = \langle t_i \mid i \in \mathbb{F}_q\rangle \cong \mathbb{Z}_p^l$ and $L = Q : \langle yc\rangle$. Then acting on $\mathrm{PG}(1, q)$, we have that $(\mathrm{PGL}(2, q))_\infty = L$ and the other points $i \in \mathrm{PG}(1, q) \setminus \{\infty\}$ correspond to the cosets Lyt_i. Since $(S : \langle z\rangle)/\langle z^2\rangle \cong \mathrm{PGL}(2, q) \cong S : \langle c\rangle$ and since z induces an automorphism of order 2 on S by conjugation, noting that $\mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q)$ has only one conjugacy class of involutions, we may assume that for any $t \in S$, the conjugacy action of z on t is given by c, that is $t^z = t^c$.

We now determine the coset graphs of $\widetilde{A}$. Let $u = B$ be the block of $\widetilde{A}$ and $\widetilde{G}$ corresponding to ∞ and let $\tilde{u} \in B$. Then $A_u \cong \widetilde{A}_{\tilde{u}} = \widetilde{G}_{\tilde{u}} \cong Q : \mathbb{Z}_{q-1}$, a Frobenius group.

Let us first determine the subgroups $\widetilde{G}_{\tilde{u}}$. Let $H_1 = Q_1 : \langle h\rangle$ be a Frobenius subgroup of $\widetilde{G}$ where $Q_1 \cong \mathbb{Z}_p^l$ and $|h| = q - 1$. Then there exists some $g \in \widetilde{G}$ such that $Q_1 K = (QK)^g = Q^g K$, that is, $Q_1 \leqslant Q^g K$. Take $1 \neq g_1 = tk \in Q_1$, where $t \in Q^g$ and $k \in K$. Then $g_1^{-1} g_1^h = (tk)^{-1}(tk)^h = t^{-1} t^h \in S \setminus \{1\}$. Therefore, $Q_1 = (g_1^{-1} g_1^h)^{\langle h\rangle} \leqslant S$, which forces $Q_1 = Q^g$. It follows that $\widetilde{G}$ has one conjugacy class of subgroups isomorphic to $\mathbb{Z}_p^l$, which are in fact contained in S. Therefore, we may assume that $Q \leqslant H := \widetilde{G}_{\tilde{u}} \leqslant N_{\widetilde{G}}(Q)$. Moreover, a direct checking shows that $N_{\widetilde{G}}(Q) = (Q : (yz))K$. Therefore, one can assume $H = Q : \langle (yz)^i z^{2s}\rangle$, for some i, j, where $|(yz)^i z^{2s}| = q - 1$. Since, in $\widetilde{G}/K$, $\overline{(yz)^i z^{2s}} = \overline{yz}^i$ must be of order $q - 1$, we know that $(i, q - 1) = 1$. Therefore, we may let $i = 1$ by rechoosing s, so that $H = Q : \langle yz^{1+2s}\rangle$. From

$$1 = \big(yzz^{2s}\big)^{q-1} = \big(yz^2 y^z\big)^{\frac{q-1}{2}} z^{2s(q-1)} = \big(yy^c\big)^{\frac{q-1}{2}} z^2 z^{2s(q-1)} = z^{(1+2s)(q-1)},$$

where noting $y^z = y^c$ as assumed above, we get

$$(1 + 2s)(q - 1) \equiv 0 \pmod{2d}. \tag{1}$$

Now the set $[\widetilde{A} : H]$ of cosets of $\widetilde{A}$ relative to H consists of two parts:

$$U(H) = \{Hk \mid k \in K\} \cup \{Hyt_i k \mid i \in \mathbb{F}_q,\ k \in K\},$$
$$W(H) = \{Hxk \mid k \in K\} \cup \{Hyt_i xk \mid i \in \mathbb{F}_q,\ k \in K\}.$$

Since there is only one edge from H to the block $\{Hyxk \mid k \in K\}$, we may assume that the neighbor of H corresponds to the bicoset $D = Hyxk'H$ for some $k' \in K$. Since the graph X is connected, we have

$$\begin{aligned}\widetilde{A} &= \langle D\rangle = \langle H, yxk'\rangle = \big\langle Q, yz^{1+2s}, yxk'\big\rangle \\ &= \big\langle \big\langle Q, Q^{yxk'}\big\rangle, yz^{1+2s}, yxk'\big\rangle = \big\langle S, yz^{1+2s}, yxk'\big\rangle = \big\langle S, z^{1+2s}\big\rangle : \langle xk'\rangle.\end{aligned}$$

Therefore, we have $\langle z^{1+2s}\rangle = \langle z\rangle$, and in particular, $(1+2s, 2d) = 1$. Consequently, by (1), we have that $q-1 \equiv 0 \pmod{2d}$, and so $d \mid \frac{q-1}{2}$ and $|yz| = q-1$.

It is easy to see that the mapping τ on $\widetilde{A}$ defined by the rule: $t \to t$ for any $t \in S$, $x \to x$, $z \to z^{-(1+2s)^{-1}}$, gives rise to an automorphism of $\widetilde{A}$ which fixes Q setwise and maps z^{1+2s} to z^{-1}. Therefore, up to graph isomorphism, we may choose so as to have $H = Q : \langle yz^{-1}\rangle$; and we let X' denote the corresponding coset graph $X(\widetilde{A}; H, D)$.

Since $\widetilde{A} = \langle D\rangle$, X' is connected. Since $|yxk'| = 2$, we have $D = D^{-1}$ and so X' is undirected. Since

$$Hyxk'\big(yz^{-1}\big) = Hy\big(yz^{-1}\big)^x xk' = Hzxk' = Hzy^{-1}yxk' = Hyxk',$$

we know that the length of the orbit of H containing the vertex $Hyxk'$ is q and so H is adjacent to Hyt_ixk' for all $i \in \mathbb{F}_q$. Therefore, the coset graph X' is a cover of Y.

Finally, we show that $X' \cong K_{q+1}^{2d}$.

For any $i \neq j \in \mathbb{F}_q$, we set $j - i = \theta^h$. Then we have that $\{Hyt_i, Hyt_jxk_1\} \in E(X')$, where $k_1 \in K$ if and only if $\{H, Hyt_jxk_1t_i^{-1}y^{-1}\} \in E(X')$. Moreover, by computation, we know that

$$\begin{aligned}
Hyt_jxk_1t_i^{-1}y^{-1} &= H\big(yt_jt_i^{-1}y^{-1}\big)xk_1 \\
&= H\big(yt_{j-i}y^{-1}\big)xk_1 = H\overline{\begin{pmatrix} 1 & 0 \\ -\theta^h & 1\end{pmatrix}}xk_1 \\
&= H\left(t_{\theta^{-h}}\overline{\begin{pmatrix} 1 & 0 \\ -\theta^h & 1\end{pmatrix}}\right)xk_1 = H\overline{\begin{pmatrix} 0 & \theta^{-h} \\ -\theta^h & 1\end{pmatrix}}xk_1 \\
&= H\overline{\begin{pmatrix} \theta^{-h} & 0 \\ 0 & \theta^h\end{pmatrix}}yt_{-\theta^{-h}}xk_1 = H\big(yy^c\big)^h yt_{-\theta^{-h}}xk_1 \\
&= H\big(yy^z\big)^h yt_{-\theta^{-h}}xk_1 \\
&= H\big(\big(yz^{-1}\big)^2z^2\big)^h yt_{-\theta^{-h}}xk_1 = Hyt_{-\theta^{-h}}xz^{-2h}k_1.
\end{aligned}$$

Since $\{H, Hyt_{-\theta^{-h}}xk'\} \in E(X')$, we get that $z^{-2h}k_1 = k'$, that is, $k_1 = (z^2)^hk'$, and so Hyt_i is adjacent to $Hyt_jx(z^2)^hk'$ (noting that $\langle z^2\rangle = K$).

Now we define a regular cover $X'' = Y \times_{f'} K$ by the following rule:

$$f'(\infty, i') = f'(i, \infty') = k', \qquad f'(i, j') = r^hk',$$

for $i \neq j \in \mathbb{F}_q$ and $j - i = \theta^h$, where $r = z^2$.

Define $\tau : V(X') \to V(X'')$ by the rule

$$\begin{aligned}
&\tau(Hk) = (\infty, k), \qquad \tau(Hyt_ik) = (i, k), \\
&\tau(Hxk) = (\infty', k), \qquad \tau(Hyt_ixk) = (i', k),
\end{aligned}$$

for any $k \in K$. It follows from the definition of the two graphs that τ is an isomorphism from the graph X' to X''.

Define $\sigma : V(X'') \to V(K_{q+1}^{2d})$ by the rule

$$\begin{aligned}
&\sigma(\infty, k) = (\infty, k), \qquad \sigma(i, k) = (i, k), \\
&\sigma(\infty', k) = \big(\infty', kk'^{-1}\big), \qquad \sigma(i', k) = \big(i', kk'^{-1}\big).
\end{aligned}$$

Then σ is an isomorphism from X'' to K_{q+1}^{2d}, and therefore $\sigma\tau$ is an isomorphism from X' to K_{q+1}^{2d}, as required.

Subcase 3.2. $S \cap K = \mathbb{Z}_2$.

In this case $S \cong \mathrm{SL}(2,q)$ and we identify S with $\mathrm{SL}(2,q)$. Then $S \cap K = \langle e \rangle$, where $e = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the unique involution of $\mathrm{SL}(2,q)$. As mentioned in the paragraph preceding Subcase 3.1, we have that $\frac{q-1}{2}$ is odd, that is, $q \equiv 3 \pmod 4$. Since $|a| = d(q+1)$, $a^2 \in SK$ and $|S \cap \langle a \rangle| = q+1$, we have that $(\frac{q+1}{2}, d) = 2$ and so $\frac{d}{2}$ is odd.

In $\mathrm{GL}(2,q)$, set

$$t_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $q = p^l$ and $i \in \mathbb{F}_q$.

Let $z_1 K$ be an involution in $(\widetilde{G}/K) \setminus (SK/K)$. Then $z_1^2 \in K = \langle r \rangle$ and, since $\frac{d}{2}$ is odd, we have the following two possibilities: if $z_1^2 = r^{2i}$, then $z := (z_1 r^{-i}) r$ is of order d; if $z_1 = r^{2i+1}$, then $z := z_1 r^{-i}$ is of order $2d$. Therefore, we may get an element z such that $zK = z_1 K$ with order d or $2d$, and $\widetilde{G} = S\langle z \rangle$. Note that acting on $\mathrm{SL}(2,q)$ by conjugation, z induces an automorphism of order 2 of $\mathrm{SL}(2,q)$. Since each automorphism τ of $\mathrm{SL}(2,q)$ has the form $\tau(t) = g^{-1} t^{\sigma} g$, where $g \in \mathrm{GL}(2,q)$ and σ is induced by the field automorphism of $\mathbb{F}_q$, and since $(\mathrm{SL}(2,q) : \langle z \rangle)/K \cong \mathrm{PGL}(2,q)$, we may assume that $t^z = t^c$. Let $u = B$ be the block of $\widetilde{A}$ and $\widetilde{G}$, corresponding to ∞ and let $\tilde{u} \in B$. Then $A_u \cong \widetilde{A}_{\tilde{u}} = \widetilde{G}_{\tilde{u}} \cong Q : \mathbb{Z}_{q-1}$. Set $H := \widetilde{G}_{\tilde{u}}$. Then, with the same argument as in Subcase 3.1, we may assume $Q \leqslant H \leqslant N_{\widetilde{G}}(Q)$. Moreover, direct checking shows that $N_{\widetilde{G}}(Q) = Q : \langle c_1, z \rangle$. Suppose that $H = Q : \langle c_1^i z^j \rangle$, where $|c_1^i z^j| = q-1$.

Let xK be an involution in $\widetilde{A}/K$ such that $[xK, gK] = K$ for any $g \in \widetilde{A}$. Then $x^2 \in K$ and $\widetilde{A} = \widetilde{G}\langle x \rangle$. With no loss of generality we may assume that $x^2 \in \langle e \rangle$, in view of the fact that $\frac{d}{2}$ is odd. Then the set $[\widetilde{A} : H]$ of coset of $\widetilde{A}$ relative to H consists of two parts:

$$\begin{aligned} U(H) &= \{Hk \mid k \in K\} \cup \{Hyt_i k \mid i \in \mathbb{F}_q,\ k \in K\}, \\ W(H) &= \{Hxk \mid k \in K\} \cup \{Hyt_i xk \mid i \in \mathbb{F}_q,\ k \in K\}. \end{aligned}$$

Now our graph must be isomorphic to a coset graph $X' = X(\widetilde{A}, H, D)$, where $D = Hyxk'H$, for $k' \in K$. Since X' is connected, we have that

$$\begin{aligned} \langle D \rangle &= \langle H, yxk' \rangle = \langle Q, c_1^i z^j, yxk' \rangle \\ &= \langle \langle Q, Q^{yxk'} \rangle, c_1^i z^j, yxk' \rangle = \langle S, c_1^i z^j, yxk' \rangle \\ &= \langle S, z^j, xk' \rangle = \widetilde{A}, \end{aligned}$$

which forces $\langle z^j \rangle = \langle z \rangle$ and so $(j, |z|) = 1$.

Note that $c_1^z = c_1^c = c_1$, $(j, |z|) = 1$ and $|c_1^i z^j| = q-1$, we have that

$$\frac{q-1}{2} = \left|\left(c_1^i z^j\right)^2\right| = \left|c_1^{2i}\right|\left|z^{2j}\right| = \left|c_1^{2i}\right|\left|z^2\right|.$$

Therefore, $|z^2|$ and $|c_1^{2i}|$ are divisors of the odd integer $\frac{q-1}{2}$, and so $|z|$ is not divisible by 4. Hence $|z| = d$ and $\widetilde{G} = S : \langle z \rangle$, where $K = \langle z^2 \rangle \times \langle e \rangle$. Since H is a Frobenius group and $z^2 \in \mathbb{Z}(\widetilde{G})$, it follows that $|c_1^i|$ is of order $(q-1)$ or $\frac{q-1}{2}$.

We claim now that $[x, S] = 1$. Actually, for any $t \in S$, we may assume $t^x = tk$ for some $k \in K$. Since S is a characteristic subgroup of $\widetilde{G}$, it follows that $t^x \in S$. This forces $k = t^{-1} t^x \in S \cap K$, and so $k^2=1$. In particular, for any elements t of odd order of S, we have $t^x=t$. Since S can be generated by all the elements of odd order, we have $[x,S]=1$.

Suppose that $z^x=zk$. Since the length of orbit of H containing $Hyxk'$ is q, we have $Hyxk'(c_1^iz^j)=Hyxk'$. Then

$$\begin{aligned}Hyxk' &= Hyxk'\big(c_1^iz^j\big)=Hy\big(c_1^iz^j\big)^{x^{-1}}xk'=Hyc_1^iz^jk^jxk'\\ &= Hz^j\big(yc_1^i\big)^{z^j}k^jxk'=Hz^j\big(yc_1^i\big)^{c^j}k^jxk'=Hz^jy^{-1}c_1^ik^jxk'\\ &= Hz^j\big(c_1^i\big)^yy^{-1}k^jxk'=Hz^jc_1^{-i}y^{-1}k^jxk'=H\big(c_1^iz^j\big)^{-1}yxk'\big(z^2k^j\big)e\\ &= Hyxk'\big(z^2ke\big)^j,\end{aligned}$$

which forces that $1=(z^2ke)^j=(z^2k)^je$. Since j is odd, $z^2ke=1$. Therefore, $z^x=zk=z^{-1}e$. In particular, since $K=\langle z^2\rangle\times\langle e\rangle$, we have that for any $k_1\in K$, $k_1^x=k_1^{-1}$.

Since X' is undirected, we have $D^{-1}=D$. Therefore,

$$\begin{aligned}(yxk')^{-1} &= k'^{-1}x^{-1}y^{-1}=x^{-1}\big(k'^{-1}\big)^{x^{-1}}y^{-1}\\ &= x^{-1}k'y^{-1}=y^{-1}x^{-1}k'=ex^{-2}yxk'\in ex^{-2}D,\end{aligned}$$

we have $ex^2=1$, that is, $x^2=e$.

Now the structure of $\widetilde{A}$ is completely determined. Let us consider the element $c_1^iz^j$ again. Suppose $|c_1^i|=\frac{q-1}{2}$. The mapping τ on $\widetilde{A}$ fixing S pointwise and x, and taking z to ze gives rise to an automorphism of $\widetilde{A}$. In this case, $\tau(c_1^iz^j)=(c_1^ie)z^j$, where $|c_1^ie|=q-1$. Therefore, we may assume that $|c_1^i|=q-1$ and moreover, one can set i=1 by rechoosing j. For the element c_1z^j, since $(\frac{q+1}{4},\frac{q-1}{2})=1$, we may choose j such that $\frac{q+1}{4}j\equiv 1 \pmod{\frac{q-1}{2}}$ for the later uses, up to graph isomorphism.

Finally, we show that $X'\cong K_{q+1}^{2d}$. For any $i\neq j\in\mathbb{F}_q$ where $q\equiv 3 \pmod 4$, we set $j-i=\theta^h$. Then we have that $\{Hyt_i,Hyt_jxk_1\}\in E(X')$, where $k_1\in K$ if and only if $\{H,Hyt_jxk_1t_i^{-1}y^{-1}\}\in E(X')$. A direct checking shows that

$$\begin{aligned}Hyt_jxk_1t_i^{-1}y^{-1} &= H\big(yt_{j-i}y^{-1}\big)xk_1\\ &= H\begin{pmatrix}1 & 0\\ -\theta^h & 1\end{pmatrix}xk_1\\ &= H\begin{pmatrix}\theta^{-h} & 0\\ 0 & \theta^h\end{pmatrix}yt_{-\theta^{-h}}xk_1\\ &= H\begin{pmatrix}(-\theta)^{-h} & 0\\ 0 & (-\theta)^h\end{pmatrix}e^hyt_{-\theta^{-h}}xk_1\\ &= H\begin{pmatrix}\theta^{-(1+\frac{q-1}{2})h} & 0\\ 0 & \theta^{(1+\frac{q-1}{2})h}\end{pmatrix}e^hyt_{-\theta^{-h}}xk_1\\ &= H\big(c_1z^j\big)^{-\frac{q+1}{2}h}\big(z^{j\frac{q+1}{2}}e\big)^hyt_{\theta^{-1}}xk_1\\ &= Hyt_{\theta^{-1}}xk_1\big(z^{-j\frac{q+1}{2}}e\big)^h\\ &= Hyt_{\theta^{-1}}xk_1\big(z^{-2}e\big)^h.\end{aligned}$$

Set $r=z^2e$. Then $K=\langle r\rangle$. Continuing the above process, we get

$$Hyt_jxk_1t_i^{-1}y^{-1} = Hyt_{\theta^{-1}}xk_1r^{-h}.$$

Since $\{H, Hyt_{-\theta^{-h}}xk'\} \in E(X')$, we get that $r^{-h}k_1 = k'$, that is $k_1 = r^hk'$ and so Hyt_i is adjacent to $Hyt_jxk'r^h$.

The rest of the proof is almost exactly the same as in Subcase 3.1, and one may show that $X' = X(\widetilde{A}, H, D) \cong K_{q+1}^{2d}$. $\square$

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